Newton's Dynamic Laws

Dr. Esmat Bekir

ABSTRACT

Introduction: "Newton's Dynamic Laws" are concerned with the motion of bodies in orbits. Newton has published his Laws in 1684, [1], and expanded this work into later editions [2]. This work was written in Latin but it was translated in many versions in English, e.g. [3]. Because of his revolutionary ideas and his mastery of geometry, his work was not largely amenable to a great sector of scientists and physicists. Several attempts, e.g. [4]-[5] were made to simplify and clarify these laws. Modern derivations may be found in e.g. [6].

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I. DYNAMIC LAWS DERIVATION

This note summarizes Newton's derivation for the planetary dynamics. I have followed [5] very closely and stripped it of all historical notes for convenience. Newton's goal, in particular, was to find out the nature of the centripetal force that causes the planet to revolve around the Sun in an elliptical orbit. He argued that, without any external influence, the planet will move in straight line at constant velocity. It is this force that imparts acceleration to the body to make it adhere to orbit.

The following figure depicts the needed graphics for analysis. It looks very crowded but just in one figure it depicts all the essential parameters and the variables of our problem.

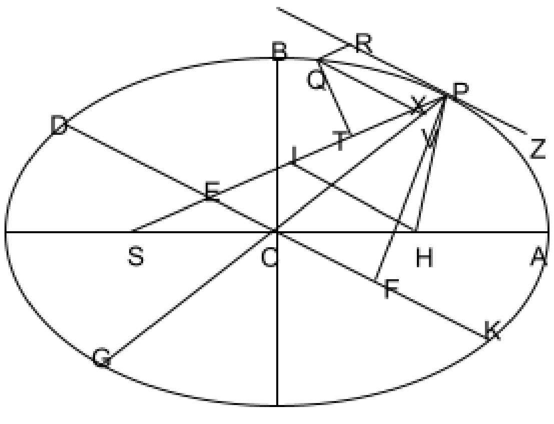

Figure 1: Based on Newton**'s** diagram for problem 3. A planet P moves in an elliptic orbit $APQB$ about a center of force S located at a focus of the ellipse.

It shows the Sun, S, resides at one focus of the ellipse. The planet P moves along the elliptical curve APQ, the tangent at P is ZPR, and the centripetal force is along PS. Newton explains that at point P, if the force vanishes, the planet will move along the tangent to R. It is the centripetal force that will move R back to Q on the ellipse.

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From Galileo's experiments, a body under the influence of central force will travel a distance proportional to the acceleration times the square of the traveled time. Conversely in time, δt, this acceleration, is proportional to QR, divided by $(\delta t)^2$.

Meanwhile, for any centripetal force, Newton proved that the line of force sweeps equal areas in equal times. This allowed him to describe the time geometrically by an area proportional to the time. That is $\delta t \propto SP \cdot QT$. Therefore, the acceleration is given by

$$A = \frac{QR}{(\delta t)^2} \propto \frac{QR}{(SP \cdot QT)^2} = \frac{QR}{SP^2 \cdot QT^2} \tag{1.1}$$

We shall analyze each term, but first introduce some preliminaries:

1. Newton's lemma:

$$PE = AC \tag{1.2}$$

   Proof is given in Appendix A.

2. ΔPXV and ΔPEC are similar, hence $PX / PE = PV / PC$

$$PX = (PE \cdot PV) / PC \tag{1.3}$$

3. Apollonius: Prop 15, Book1 $PV \cdot GV / QV^2 = PC^2 / CD^2$

$$PV = (QV^2 \cdot PC^2)/(CD^2 \cdot GV) \tag{1.4}$$

Aside from the proofs given in [7], Appendix B provides a little simple proof.

4. The shape QRPX is quadrilateral,

$$\square QRPX \Rightarrow QR = PX \tag{1.5}$$

   Thus, the last four equations yield,

$$QR = PX = (PE \cdot PV) / PC = (AC \cdot QV^2 \cdot PC)/(CD^2 \cdot GV) \tag{1.6}$$

5. Apollonius: Prop 31, Book7

$$CA \cdot CB = CD \cdot PF \tag{1.7}$$

   Proof is given in Appendix C.

6. ΔQTX and ΔPFE are similar, hence

$$QT / QX = PF / PE \tag{1.8}$$

Substituting from Eqs. (1.2) and (1.7) into (1.8) gives

$$\begin{aligned} &QT / QX = (CA \cdot CB)/(CA \cdot CD) \Rightarrow \\ &QT = (QX \cdot CB) / CD \end{aligned} \tag{1.9}$$

Now, Eqs. (1.6) and (1.9) yield

$$\begin{aligned} \frac{QR}{QT^2} &= (AC \cdot QV^2 \cdot PC \cdot CD^2)/(CD^2 \cdot GV \cdot CB^2 \cdot QX^2) \\ &= (AC / CB^2) \cdot (PC / GV) \cdot (QV^2 / QX^2) \end{aligned} \tag{1.10}$$

In the limit Q approaches P. Consequently V and X will approach P. That is

$$GV = 2PC$$
$$QX = QV \tag{1.11}$$

Using the above equation in Eq. (1.10) gives

$$\frac{QR}{QT^2} = (2/L)\cdot(1/2) = 1/L \tag{1.12}$$

Here, $L = CB^2/(2\cdot AC)$ is the latus rectum of the ellipse. Finally, Eq. (1.1) results in,

$$A = \frac{QR}{(\delta t)^2} \propto \frac{QR}{(SP\cdot QT)^2} = \frac{(1/L)}{SP^2} \propto \frac{1}{SP^2} \tag{1.13}$$

In words, the centripetal force is inversely proportional to the Sun/Planet radius squared.

Kepler's Third Law

Newton then gave a proof to Kepler's third law – the orbit's period is proportional to one and half the power of the ellipse major axis diameter. Herein, we provide a proof that is a little variant from Newton's.

Newton has shown that for any centripetal force, a planet will sweep an area proportional to time. Thus from the diagram, the area swept for the infinitesimal time, δt, is

$$Swept\ area\ in\ \delta t = SP\cdot QT \Rightarrow$$
$$rate\ of\ Swept\ area = SP\cdot QT/\delta t \tag{1.14}$$

Since ellipse area $= \pi\cdot CB\cdot AC$, then the time period T is

$$T = \pi\cdot CB\cdot AC/(SP\cdot QT/\delta t) = \pi\cdot CB\cdot AC\cdot\delta t/(SP\cdot QT) \tag{1.15}$$

Squaring each side of Eq. (1.15) yields,

$$T^2 = \pi^2\cdot CB^2\cdot AC^2\cdot\delta t^2/(SP^2\cdot QT^2) \tag{1.16}$$

Substituting for QT from Eq.(1.12) into the above gives

$$T^2 = \pi^2\cdot CB^2\cdot AC^2\cdot\delta t^2/(SP^2\cdot L\cdot QR) \tag{1.17}$$

Since $L = CB^2/(2\cdot AC)$, the above can be simplified to

$$T^2 = AC^3\cdot\left[2\pi^2\cdot\delta t^2/(SP^2\cdot QR)\right] \tag{1.18}$$

Equation (1.13) implies $QR\cdot SP^2 \propto (\delta t)^2$ and therefore, makes (1.18)

$$T^2 \propto AC^3 \tag{1.19}$$

That is the orbit's period squared is proportional to the cube of the ellipse semi axis diameter.

Appendix A

Newton's Lemma
PE=AC

Proof: H is the 2nd foci of the ellipse from which we draw HI parallel to RPZ. Ellipse reflection property, [8], states $\angle HZP = \angle SPR$. Therefore, $\angle PHI = \angle HZP = \angle SPR = \angle HIP$. Hence ΔHIP is equal sided triangle and $PH=PI$.
Moreover:

$$PH + PS = 2AC \Rightarrow$$
$$PI + (PI + EI + SE) = 2PI + (EI + SE) = 2AC \quad \text{(A.1)}$$

Since HI parallel to RPZ and thus to DK and since $SC=CH$, then $SE=EI$. Therefore

$$2PI + 2EI = 2PE = 2AC \quad \text{(A.2)}$$

or,

$$PE = AC \quad \text{(A.3)}$$

Appendix B

Chord Bisector
Apollonius Proposition 15

The eccentric circle is that one that shares the ellipse's major axis. Herein, we project the ellipse points P, Q, D, G and K onto P', Q', D', G' and K' on the eccentric circle. Consequently, the lines $P'CG'$ and $D'CK'$ become two perpendicular diameters in the eccentric circle. The point V on the CP is mapped onto V' on the line $P'Q'$ so that $Q'V'$ is perpendicular to the diameter $P'CG'$, hence

$$V'P' \cdot G'V' = Q'V'^2 \quad \text{(B.1)}$$

Notice that $\Delta CVV'$ is similar to $\Delta CPP'$, thus

$$\frac{VP}{V'P'} = \frac{CP}{CP'} \quad \text{(B.2)}$$

Likewise $\Delta CGG'$ is similar to $\Delta CPP'$, thus

$$\frac{VG}{V'G'} = \frac{CP}{CP'} \quad \text{(B.3)}$$

The above two equations give

$$\frac{VP \cdot VG}{V'P' \cdot V'G'} = \frac{CP^2}{CP'^2} \quad \text{(B.4)}$$

Substituting from (B.1) yields

$$VP \cdot VG = Q'V'^2 \frac{CP^2}{CP'^2} \quad \text{(B.5)}$$

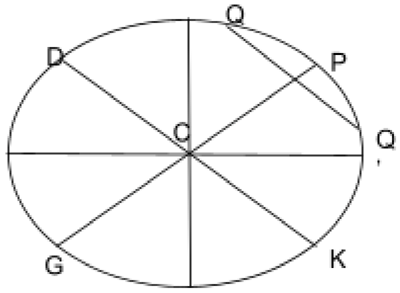

Figure 2a: The diameter PG bisects the chord QQ' and DK. From Proposition 15 of Book 1 of *Apollonius's Conics*, the ratio of $PVxVG/QV^2$ equals the ratio PC^2/DC^2.

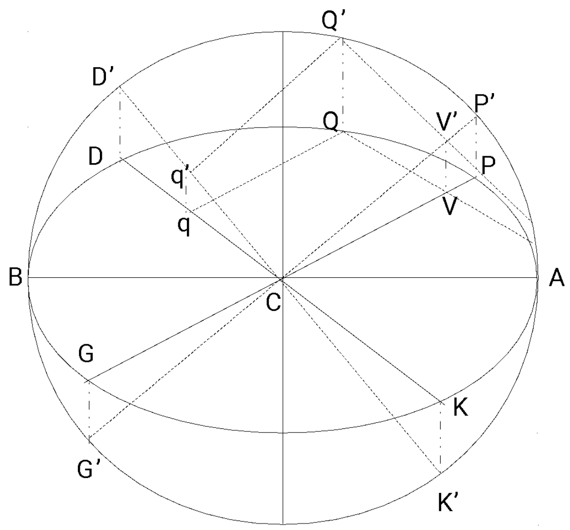

Figure 2b: projecting the ellipse points P, Q, D, G and K onto P', Q', D', G' and K' on the eccentric circle.

What remains is to relate QV to $Q'V'$. We know that QV is parallel to DC, so from Q we draw a parallel to PC until it meets DC in q, therefore $Cq=QV$. Project q onto point q' on $D'C$, and it is straight forward to prove that $C\,q'=Q'V'$. Now $\Delta CDD'$ is similar to $\Delta Cqq'$, Thus

$$\frac{V'Q'}{VQ}=\frac{Cq'}{Cq}=\frac{CD'}{CD} \tag{B.6}$$

Substituting in (B.5) results in

$$VP\cdot VG=QV^2\frac{CD'^2}{CD^2}\frac{CP^2}{CP'^2} \tag{B.7}$$

But $CP'=CD'=a$ implies that

$$VP\cdot VG=QV^2\frac{CP^2}{CD^2} \tag{B.8}$$

Appendix C

Parallelogram area Equivalence
Apollonius Proposition 31 Book 7

With reference to Fig. 3 below, we introduce two proofs.

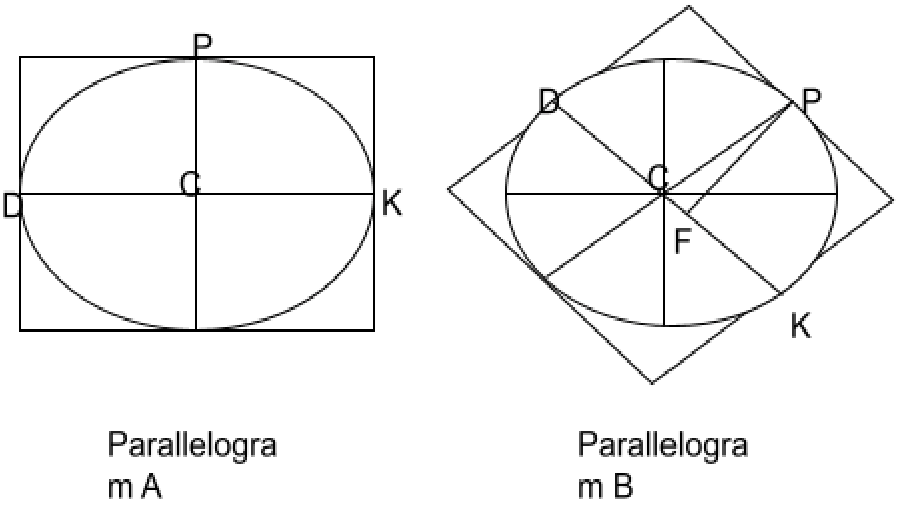

Figure 3a: Area of parallelogram A is equal to area of parallelogram B (Proposition 31, Book 7, of the Conics of Apollonius of Perga).

Method 1: geometric algebra

Let *P'* and *D'* be the points on the eccentric circle from which their projection *P* and *D* on the ellipse are obtained. The coordinates of $\bar{P}$ and $\bar{D}$ are then given by

$$x_p = x_1 \quad y_p = \frac{a}{b}y_1 = \frac{am_1}{b}x_1 \tag{C.1}$$

and,

$$x_d = x_2 = \frac{a}{b}y_1 == \frac{am_1}{b}x_1 = y_p \qquad y_d = \frac{a}{b}y_2 = x_1 \tag{C.2}$$

Since $x_d = y_p$ and $y_d = x_p$ then *CP'* and *CD'* are perpendicular and thus the area enclosed by *CP'* and *CD'* $= CK^2 = a^2$. Therefore the area enclosed by CP and $CD = \frac{b}{a}CK^2 = ab$ as desired.

Method 2: vector analysis: In Fig. 3b, let *P* and *D* denote the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively. The area *PCD* is then given by the cross product of vectors *CP* and *CD*. Thus

$$\begin{Vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{Vmatrix} = x_1y_2 - x_2y_1 = x_1\frac{b}{a}x_1 + \frac{a}{b}m_1x_1m_1x_1 = \frac{1}{ab}x_1^2(b^2 + a^2m_1^2)$$

$$= \frac{a^2b^2}{ab} = ab \qquad Q.E.D$$

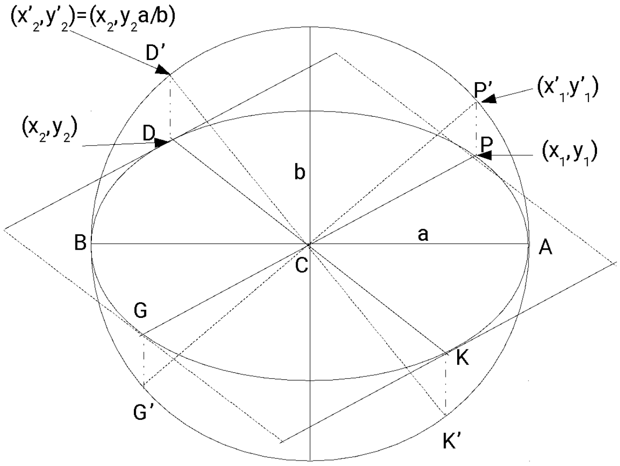

Figure 3b: Projecting the ellipse points *P, D, G* and *K* onto *P'D', G'* and *K'* on the eccentric circle.

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